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PERIODIC MOTIONS OF A HEAVY, RIGID, DYNAMICALLY ALMOST SYMMETRIC BODY WITH A FIXED POINT*

V.S. SERGEEV

A motion is studied, about a fixed point, of a heavy rigid body, the mass distribution in which resembles the mass distribution encountered in the Lagrange's case. The equations of motion are written in the action-angle variables for the Lagrange's case, after which the method developed by Poincaré /1/ for such systems is used to prove the existence of new families of periodic motions represented by series in powers of a small parameter introduced into the process.

Periodic motions of a reduced system were studied under the above conditions in /2,3/. The action-angle variables were used in the Euler-Poinsot problem in /4/ and employed in /5/ to prove the existence of periodic motions of general type in the corresponding perturbed problem.

In the Lagrange's case the action-angle variables were used for a system with excluded cyclic coordinate in /6/ to clarify the problem of existence of the conditionally periodic motions of a rigid body differing little from the Lagrange gyroscope. The action-angle variables were used in /7/ in the Lagrange's case for the complete system, where the expansions into Fourier series of the direction cosines of the vertical were also given.

1. We shall describe the motion of a heavy rigid body using the Euler angles ϑ, φ, ψ and the conjugated canonic impulses $p_\vartheta, p_\varphi, p_\psi$. We introduce the small parameter $\mu \geqslant 0$ by means of the formulas

$$B = A\,(1 + \mu D)^{-1},\ \ x_0 = \mu x_1 l,\ \ y_0 = \mu y_1 l,\ \ z_0 = z_1 l > 0$$

containing the moments of inertia A, B principal for the fixed point O, the coordinates x_0, y_0, z_0 of the center of mass of the body relative to the coordinate system attached to the principle axes of inertia of the point O, the dimensionless quantities D, x_1, y_1, z_1 which are assumed finite compared with the small parameter μ, and the characteristic dimension of the body l. We assume that the O_z axis is fixed in the space of the $Oxyz$ coordinate system and directed vertically downwards. We now pass to the new variables $I = (I_1, I_2, I_3)$, $w = (w_1, w_2, w_3)$ which are the action-angle variables in the Lagrange's case. We assume that the roots e_i ($i = 1,2,3$) of the known third order equation in $\cos\vartheta$ defining the domain of admissible values of ϑ satisfy the inequalities

$$e_3 < -1,\ \ -1 < e_2 < e_1 < 1 \tag{1.1}$$

Then the transformation will have the form

$$\begin{gathered}
p_\vartheta = \left[\beta + \beta_0\cos\vartheta - \frac{(I_2 - I_3\cos\vartheta)^2}{\sin^2\vartheta}\right]^{1/2},\quad p_\psi = I_2,\quad p_\varphi = I_3 \\
\cos\vartheta = e_1\,\mathrm{cn}^2\left(\frac{K}{\pi}w_1\right) + e_2\,\mathrm{sn}^2\left(\frac{K}{\pi}w_1\right),\quad w_2 = \psi - \chi_+,\quad w_3 = \varphi + \chi_- \\
\chi_\pm = (\sqrt{\beta_0}\,\varepsilon_3 K)^{-1}\left(Q_\pm(\alpha)\,K - Q_\pm\left(\frac{\pi}{2}\right)F(\alpha, k)\right) \\
Q_\pm(\alpha) = \frac{I_2 - I_3}{1 - e_1}\,\Pi(\alpha, b_+, k) \pm \frac{I_2 + I_3}{1 + e_1}\,\Pi(\alpha, b_-, k) \\
\sin\alpha = \sqrt{e_1 - \cos\vartheta}/\varepsilon_2,\quad \varepsilon_2^2 = e_1 - e_2,\quad \varepsilon_3^2 = e_1 - e_3 \\
\beta_0 = 2MgAz_0,\quad \beta = (2k_1 - I_3^2/C)A,\quad b_\pm = \pm\varepsilon_2^2/(1 \mp e_1)
\end{gathered} \tag{1.2}$$

Here $k = \varepsilon_2/\varepsilon_3$ is the modulus of the elliptic integrals and functions, Mg is the weight of the body, C is the principal moment of inertia and k_1 is the constant of the area integral. The quantities β, e_i in formulas (1.2) are assumed to be expressed in terms of I. When $\mu = 0$, the variables w_i vary, in the general case, in the conditionally periodic manner, with the frequencies

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$$\omega_1 = \frac{\pi\varepsilon_3\sqrt{\beta_0}}{2AK}, \quad \omega_2 = \frac{Q_+(\pi/2)}{2AK}, \quad \omega_3 = I_3\frac{A-C}{AC} - \frac{Q_-(\pi/2)}{2AK}$$

2. Let us write the Hamiltonian function of the problem in the form

$$H = H_0(I) + \mu H_1(I, w_1, w_3), \quad H_1(I, w_1, w_3) = \tag{2.1}$$
$$\frac{D}{2A}\left[\frac{\cos\varphi}{\sin\vartheta}(I_2 - I_3\cos\vartheta) - p_\vartheta\right]^2 - Mgl(x_1\sin\varphi + y_1\cos\varphi)\sin\vartheta$$

where $\vartheta, \varphi, p_\vartheta$ depends on I and w in accordance with the formulas (1.2). Using the Poincaré method /1/ we shall seek the periodic solutions of the system of equations with Hamiltonian (2.1) similar to periodic solutions of the unperturbed integrable problem. We assume that the relations $\omega_k T = 2\pi m_k,\ k = 1, 2, 3$ where $m_k \neq 0$ hold for some $T > 0$. Then the general solution of the system in question where

$$I_k = a_k, \quad w_k = \omega_k t + \alpha_k \tag{2.2}$$

and a_k, α_k are constants, will be T-periodic. Using the Poincaré theorem we can assert that at small μ at least two families of T-periodic solutions will exist, provided that the following conditions hold:

$$\frac{\partial^2[H_1]}{\partial\alpha_3^2} \neq 0 \text{ when } \frac{\partial[H_1]}{\partial\alpha_3} = 0 \tag{2.3}$$

where $[H_1]$ denotes the function H_1 averaged over the period T, into which the solution (2.2) has been substituted, and

$$\delta = \frac{D(\omega_1, \omega_2, \omega_3)}{D(I_1, I_2, I_3)} \neq 0 \tag{2.4}$$

where $I_k = a_k$. The above periodic solutions are holomorphic functions of parameter μ.

Let us turn our attention to condition (2.4). Let δ_1 denote the Jacobian of ω_k in e_j and δ_2 the Jacobian e_j over $I_k\ (j, k = 1, 2, 3)$, in which case we have $\delta = \delta_1\delta_2$, and we have the following expression for δ_2:

$$\delta_2 = \frac{4\pi\varepsilon_3\beta_0^{3/2}K(I_3^2 - I_2^2)}{(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)} \tag{2.5}$$

We note that under the assumptions that (1.1) the inequality $|I_2| \neq |I_3|$ holds and therefore according to (2.5) $\delta_2 \neq 0$. It can be shown, e.g. by computing the first terms of the expansion of δ_2 into a series in powers of k^2, that the Jacobian $\delta_1 \not\equiv 0$. Here δ_1 is of the order of k^2 and can be written in the form

$$\delta_1 = \frac{k^2\beta_0^{1/2}}{64A^3C\varepsilon_3(1 - e_1^2)(1 - e_3^2)}[3As_2(e_3s_2 - s_3) - 4C\varepsilon_3^2(I_2^2 - I_3^2)] + \tag{2.6}$$
$$k^4\Delta_1(k^2), \quad s_2 = I_2 - e_1I_3, \quad s_3 = I_3 - e_1I_2$$

where $\Delta_1(k^2)$ is a holomorphic function of k^2. The expression within the square brackets in (2.6) is not identically zero, and this can be confirmed by expressing the variables I_2 and I_3 in terms of the independent quantities $\beta_0, e_1, e_3\ (e_2 = e_1)$.

Next we consider the condition (2.3) which can also be fulfilled. Indeed, writing for simplicity $A = B$ and assuming from the opposite that the relations $\partial[H_1]/\partial\alpha_3 = 0$ and $\partial^2[H_1]/\partial\alpha_3^2 = 0$ are satisfied simultaneously, we find, as $x_1^2 + y_1^2 \neq 0$, that

$$\int_0^T \exp[i(\Phi_1 + \omega_3 t + \alpha_3)]\sin\vartheta\, dt = 0 \tag{2.7}$$

Here the arguments in the functions $\Phi_1 = \varphi - w_3, \vartheta$ depending in accordance with (1.2) on I and w_1, have been changed according to (2.2). After transforming (2.7) we arrive at the relations

$$\int_0^T \exp(i\omega_3 t) R\, dt = 0 \tag{2.8}$$
$$R = \cos\Phi_1\sin\vartheta - \omega_3^{-1}\frac{dS}{dt}, \quad S = \sin\Phi_1\sin\vartheta$$

The function R periodic in t with period $T_1 = 2\pi/\omega_1$, can be expanded on the segment $[0, T_1]$ into a Fourier series the coefficients of which are denoted by r_n. The integral (2.8) is not zero only when $\omega_3 = n\omega_1$ ($n \neq 0$ is an integer) and in this case it is equal to $r_n T$. Thus for every n from the set of values such that $r_n \neq 0$ and $\omega_3 = n\omega_1$ and provided that (2.4) holds, there exist solutions to the problem periodic in $\vartheta, \psi, \varphi \pmod{2\pi}$. We shall show that the set of values of n in nonempty. To do this we expand with help of (1.2) the function R, for $k^2 \ll 1$, into a Fourier series, and write the first terms of the expansion in powers of k^2 thus

$$R = \sqrt{1-e_1^2}\left[1 + \frac{k^2 e_1 \varepsilon_3^2}{1-e_1^2}\sin^2\frac{w_1}{2} - \right. \\ \left. \frac{k^2\varepsilon_3}{4\sqrt{\beta_0}}\left(\frac{I_2 - I_3}{(1-e_1)^2} + \frac{I_2 + I_3}{(1+e_1)^2}\right)\cos w_1 + k^4 R_2(w_1, k^2)\right. \tag{2.9}$$

where $R_1(w_1, k^2)$ is a holomorphic function of k^2. From (2.9) it follows that periodic solutions exist for which $\omega_1 = \omega_3$. The function $R_1(w_1, 0)$ contains a harmonic of $\cos 2w_1$, therefore periodic motions with $\omega_3 = 2\omega_1$ must also exist and we can have $\omega_3 = (m_3/m_2)\omega_2$. The existence of periodic solutions for $n > 2$ follows from the results of the paper (*).

3. We shall show that families of solutions exist at small μ, describing periodic motions with periods close to those of periodic motions of the unperturbed problem. We introduce a new independent variable τ by means of the formula

$$t = (1+\alpha)\tau \tag{3.1}$$

where $\alpha = \mu\alpha'$ and α' is a holomorphic function of μ. We shall seek periodic solutions of the Hamiltonian system of equations transformed according to (3.1), T-periodic in τ and satisfying at $\tau = 0$ the condition

$$I_k = a_k + \beta_k, \quad w_k = \alpha_k + \beta_{k+3}, \quad k = 1, 2, 3$$

where $\beta_k = \mu\beta_k'$ and β_k' are holomorphic functions of μ, to be defined. Then the conditions of periodicity of the functions assume the form

$$\mu\left(\omega_i \alpha' T + \sum_{k=1}^{3}\frac{\partial\omega_i}{\partial I_k}\beta_k' T + \int_0^T \frac{\partial H_1}{\partial I_i}d\tau\right) + \mu^2\Psi_i = 0 \tag{3.2}$$

where Ψ_i are holomorphic functions of μ, α', β_i. We shall regard the relations (3.2) as equations connecting α', β_k' ($k = 1, 2, 3$). The sufficient condition of their solvability with respect to e.g. $\alpha', \beta_1', \beta_2'$ is, that the inequality

$$\delta_3 = \begin{vmatrix} \omega_1 & \omega_{1,1}' & \omega_{1,2}' \\ \omega_2 & \omega_{2,1}' & \omega_{2,2}' \\ \omega_3 & \omega_{3,1}' & \omega_{3,2}' \end{vmatrix} \neq 0 \quad \left(\omega_{i,k} = \frac{\partial\omega_i}{\partial I_k}\right) \tag{3.3}$$

holds for $I_k = a_k$. The initial value of the variable I_3 remains arbitrary. Thus the fulfilment of the inequalities (3.3) and (2.3) is a sufficient reason for the existence of two families of periodic motions, with the period, determined uniquely from initial conditions in accordance with (3.2), is close to the period of the periodic motion of the unperturbed problem.

Let us compute the determinant δ_3 (3.3) for $k^2 \ll 1$, expressing it in terms of the Jacobian

$$D(\omega_1, \omega_2, I_3\omega_3)/D(I_1, I_2, I_3)$$

and Jacobians δ_1, δ_2. Using (2.6) we obtain

$$\delta_3 = \frac{k^2\beta_0^{1/2}\delta_2}{64A^2C\varepsilon_3(1-e_1^2)(1-e_3^2)}\left\{\frac{3I_3 s_2}{1-e_1^2}(A-2C)(e_3 s_2 - s_3) + \right. \\ (I_2^2 + I_3^2)\left[\frac{6I_3 e_3^2}{1-e_1^2}(2A - C) - \frac{Cs_2}{1-e_1^2} + \frac{s_1}{1-e_3^2} + \right. \\ \left.\left. \frac{4}{(1-e_1^2)^2}(s_2(1-e_1e_3) - s_3\varepsilon_3^2)\right]\right\} + k^4\Delta_3(k^2), \quad s_1 = I_2 - e_3 I_3 \tag{3.4}$$

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*) Sergeev V.S. On periodic motions of a heavy rigid body rotating about a fixed point. Preprint VTs Akad. Nauk SSSR, Moscow, 1981.

where $\Delta_3(k^2)$ is a holomorphic function of k^2. From (3.4) we see that for small $k^2, \delta_3 \not\equiv 0$, and therefore the periodic motions sought exist. Using (3.4) we can obtain from (3.2) an explicit expression for the principal term of the expansion of α' into a series in μ at small k^2.

The families of periodic motions obtained $T(1+\alpha)$-periodic in t depend on four arbitrary initial conditions. Every periodic solution corresponding to the periodic motions shown, has at least four characteristic zero indices /1/. If the condition of isoenergetic non-degeneracy is fulfilled for the reduced system, then one of the families of periodic motions the remaining two characteristic indices are real and have opposite signs /1,5/, i.e. the motions of this family are unstable. The second family has in this case, in addition to the zero indices, two purely imaginary characteristic indices /1,5/.

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